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ON THE OPTICAL TRANSFORMATION OF COORDINATES AFTER TWO REFLECTIONS

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Abstract—A need for an optical transformation of coordinates arises in a number of problems of optical data processing, adaptive optics and reflecting antenna theory. For a plane element with (or without) a lens, this problem has been tackled in the paraxial approximation. This work demonstrates that, when the optical medium consists of two mirrors, under certain conditions the transformation problem has a solution outside the paraxial region. Explicit mirror equations are derived for the mirrors that realize any given transformation of coordinates. Application to reflecting antenna theory is also discussed.

1. INTRODUCTION

In solving the problems of optical data processing, adaptive optics, reflector-type antennae and optical instrumentation for astronomy, there is a need for an optical transformation of coordinates in which a ray with coordinates (x, y) in the input plane having traversed some optical medium (passive or adaptive) is transformed into a ray with coordinates

$$u = u(x, y), \qquad v = v(x, y) \tag{1}$$

in the output plane of the optical system. For a combination of a plane optical element with a lens, this problem has been tackled in the paraxial approximation [1, 2] and in the Fresnel region (i.e. without a lens) [2]. These authors have obtained a necessary and sufficient condition for a smooth phase function of an optical element to exist, *viz.*

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}. \tag{2}$$

The present paper is intended to prove that for an optical medium consisting of two smooth reflecting mirrors this same condition is necessary and sufficient for resolvability outside the paraxial region also. The incident and reflected wavefronts are assumed to be planar.

2. DERIVATION OF EQUATIONS

The problem geometry is obvious from Fig. 1. For simplicity assume that the wave travels in the same direction after two reflections. We assume also that the phase fronts of the incident and reflected waves with coordinates (x, y) and (u, v), respectively, coincide with the XOY plane of the Cartesian coordinate system. Having undergone two reflections, a ray with coordinates (x, y) becomes a ray with coordinates (u, v). The relevant laws of geometrical optics are those of constant optical path and reflection at the mirrors. Let us represent them in a form convenient for analysis. The first of them is

$$z + z_1 + \sqrt{(x-u)^2 + (y-v)^2 + (z+z_1)^2} = m, \tag{3}$$

where $m = \text{const.}$ is the optical path length, and $z = z(x, y)$ and $z_1 = z_1(u, v)$ are the equations of the reflecting surfaces. Rewrite Eq. (3) as

$$2m(z + z_1) + (x-u)^2 + (y-v)^2 - m^2 = 0, \tag{4}$$

and denote its left-hand side by H. A straightforward argument indicates that the reflection law at the mirror $z = z(x, y)$ may be written as

$$\frac{\partial H}{\partial x} + \frac{\partial H}{\partial z}\frac{\partial z}{\partial x} = 0, \qquad \frac{\partial H}{\partial y} + \frac{\partial H}{\partial z}\frac{\partial z}{\partial y} = 0,$$

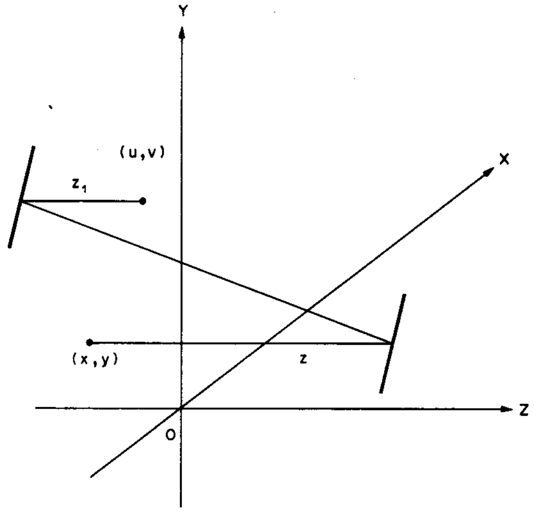

Fig. 1.

or

$$\frac{\partial z}{\partial x} = -\frac{\partial H}{\partial x} : \frac{\partial H}{\partial z} = -\frac{x-u}{m},$$

$$\frac{\partial z}{\partial y} = -\frac{\partial H}{\partial y} : \frac{\partial H}{\partial z} = -\frac{y-v}{m}.$$

The law of reflection at the second mirror follows from the reflection law at the first and Eq. (4). Since

$$\frac{\partial^2 z}{\partial z \, \partial y} = \frac{\partial^2 z}{\partial y \, \partial x}, \tag{5}$$

then from Eq. (5) there follows Eq. (2).

Thus, the transformation of coordinates $u = u(x, y)$, $v = v(x, y)$ is implementable with the aid of two reflections provided only that Eq. (2) holds.

Let (x, y) vary in a certain connected domain D. Then, by the known theorem of analysis, if condition (2) is valid then in the domain D there exists a function $T(x, y)$ such that

$$\mathrm{d}T = u(x, y)\,\mathrm{d}x + v(x, y)\,\mathrm{d}y \tag{6}$$

or

$$T(x, y) = T(x_0, y_0) + \int_C u(x, y)\,\mathrm{d}x + v(x, y)\,\mathrm{d}y, \tag{7}$$

where the integral is taken over any curve lying in D and connecting the points (x, y) and (x_0, y_0). From (5) there follows

$$\mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y = -\frac{1}{m}(x\,\mathrm{d}x + y\,\mathrm{d}y) + \mathrm{d}T.$$

Integration of this equation yields

$$z(x, y) = T(x, y) - \frac{x^2 - y^2}{2m} + \frac{x_0^2 + y_0^2}{2m} - T(x_0, y_0), \tag{8}$$

and the function $z_1(x, y)$ can be found by Eq. (4).

3. SPECIFIC CASES OF TRANSFORMATION

The class of transformations of coordinates satisfying (2) is rather wide. Some specific cases of this class will be given below.

3.1. *Conformal transformation*

Augmenting (2) by the second equation of the Cauchy–Riemann set

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$$

we obtain that any conformal mapping of a plane can be effected by means of two reflections.

3.2. *Transformation with separable variables*

Let (1) have the form

$$\begin{aligned} u &= f_1(x)F_2(y), \\ v &= F_1(x)f_2(y). \end{aligned} \tag{9}$$

Substituting in (2) yields

$$f_1(x)/F_1'(x) = f_2(y)/F_2'(y) = c,$$

where c is the separation constant. Substituting this in (9) we obtain the general form of transformation of coordinates with separable variables, which can be realized with the aid of two mirrors

$$\begin{aligned} u &= cF_1'(x)F_2(y), \\ v &= cF_1(x)F_2'(y), \end{aligned}$$

where F_1 and F_2 are arbitrary functions satisfying the inversibility condition (9), namely,

$$F_1(x)F_1''(x)F_2(y)F_2''(y) - (F_1'(x)F_2''(y))^2 \neq 0.$$

The mirror equation $z(x, y)$ may be explicitly found from Eq. (8) as

$$z(x, y) = cF_1(x)F_2(y) - \frac{x^2 + y^2}{2m} + \frac{x_0^2 + y_0^2}{2m} - cF_1(x_0)F_2(y_0),$$

where (x_0, y_0) is an arbitrary point in D.

3.3. *A group of affine transformations*

$$\begin{aligned} u &= a_1x + b_1y + c_1, \\ v &= a_2x + b_2y + c_2. \end{aligned} \tag{10}$$

Condition (2) implies that $b_1 = a_2$, i.e. Eqs (10) represent a symmetric linear transformation. A more general transformation

$$\begin{aligned} u &= f_1(x) + by + c_1, \\ v &= bx + f_2(y) + c_2, \end{aligned}$$

satisfying (2) may also be considered.

3.4. *Transformation of polar coordinates*

If the polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$ are introduced at the initial wavefront, then the condition (2) takes the form

$$\frac{\partial u}{\partial \varphi}\cos\varphi + \frac{\partial u}{\partial r} r\sin\varphi = \frac{\partial v}{\partial r} r\cos\varphi - \frac{\partial v}{\partial \varphi}\sin\varphi.$$

If (1) allows separation of variables in the form

$$u = f_1(r)\cos\varphi, \qquad v = f_2(r)\sin\varphi,$$

then by substituting this in (10) we find that $f_2(r) = f_1 + cr$, c being an arbitrary constant.

4. TWO-MIRROR ANTENNA PROBLEM

In order to obtain a desired radiation pattern in the synthesis of reflecting antennae, one must also ensure a specified, say constant, distribution of amplitude in the antenna aperture. Mathematically this implies that the transformation (1) should preserve some integral invariant. To make it more specific, if the intensities of the incident and reflected waves are $I(x, y)$ and $G(u, v)$, then by conservation of energy $I(x, y)\,dx\,dy = G(u, v)\,du\,dv$ or

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \pm\frac{I(x, y)}{G(u, v)}. \tag{12}$$

This equation taken together with (2) forms a system of differential equations for u and v. This system may be reduced to one equation with the help of the function $T(x, y)$. Indeed, since

$$\frac{\partial T}{\partial x} = u(x, y), \qquad \frac{\partial T}{\partial y} = v(x, y),$$

then by differentiating again with respect to x and y and substituting in (12) we get

$$\frac{\partial^2 T}{\partial x^2}\frac{\partial^2 T}{\partial y^2} - \left(\frac{\partial^2 T}{\partial x\,\partial y}\right)^2 = \pm\frac{I(x, y)}{G\left(\dfrac{\partial T}{\partial x}, \dfrac{\partial T}{\partial y}\right)}. \tag{13}$$

This is a Monge–Ampere equation. The plus sign relates to the elliptic type, and the minus sign to the hyperbolic type.

5. SATISFYING BOUNDARY CONDITIONS

Suppose that the two singly connected domains D_1 and D_2 for which the bounds $K_1 = (x(t), y(t))$ and $K_2 = (u(\tau), v(\tau))$ are given on the incident and reflected wavefronts, respectively. We seek to answer the question of whether there exist two mirrors mapping D_1 into D_2 which transform K_1 into K_2. By the Riemann theorem on conformal transformation [3] we answer this question in the affirmative if the boundaries of these domains are piecewise smooth. In particular, a circle may be transformed into a square. Finally, if we additionally impose a required amplitude distribution on D_1 and D_2, then the required transformation can be obtained by solving the Neumann type problem for Eq. (13).

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